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Dynamics of Quadratic Stochastic Operators with Some Coordinates Invariant

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Abstract

In this article, we determine all fixed points of quadratic stochastic operators with invariant coordinates, and we conduct a comprehensive analysis of the local and global dynamics of the given operator. In particular, we investigate the stability of the fixed points and classify their types using appropriate dynamical system techniques. Furthermore, we study the asymptotic behavior of trajectories generated by the operator and establish conditions under which convergence occurs. Several illustrative examples are provided to demonstrate the theoretical results and to highlight different dynamical regimes, including stability, periodicity, and convergence to boundary points.

Keywords: Quadratic stochastic operator, Fixed point, Simplex, Trajectory.

1|Introduction

One of the most actively developing directions of modern mathematics is the theory of dynamical systems. This theory studies the evolution of processes over time and is investigated in both discrete and continuous settings. Dynamical systems are widely applied in many fields such as biology, economics, physics, and information technology. Scientists such as Poincare, Lyapunov, and Kolmogorov played an important role in the formation of this theory.

The Uzbek mathematical school has also achieved significant results in this direction. In particular, stochastic operators, dynamical processes on simplices, and their stability properties have been deeply studied by T.A. Sarimsoqov, R.A. Ganikhodzhaev, N.N. Ganikhodzhaev, U.A. Rozikov, and U. Jamilov.

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Let

$$S^{n-1} = \left\{ \mathbf{x} \in \mathbb{R}^n : \ x_i \geq 0, \ i = \overline{1,n}, \ \sum_{i=1}^{n} x_i = 1 \right\}$$

be the $(n-1)$-dimensional simplex, and define an operator

$$V : S^{n-1} \to S^{n-1}$$

by

$$x_k' = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j, \qquad k = 1, \dots, n. \tag{1}$$

Here,

$$P_{ij,k} = P_{ji,k} \geq 0, \qquad \sum_{k=1}^{n} P_{ij,k} = 1. \tag{2}$$

Definition 1. *The operator defined by 1-2 is called a* quadratic stochastic operator *(QSO).*

The condition

$$P_{ij,k} = 0, \quad k \notin \{i, j\} \tag{3}$$

is called the *Volterra condition*, while

$$P_{ij,k} = 0, \quad k \in \{i, j\} \tag{4}$$

is called the *non-Volterra condition.* If 3 holds, then the operator is called a Volterra QSO; if 4 holds, it is called a non-Volterra QSO.

Definition 2. *Let $V : S^{n-1} \to S^{n-1}$ be a quadratic stochastic operator. A point $x^* \in S^{n-1}$ is called a* fixed point *of V if*

$$V(x^*) = x^*.$$

We define a quadratic stochastic operator $V : S^2 \to S^2$ as follows:

$$\begin{aligned} x_k' &= \sum_{i,j=1}^{3} P_{ij,k} x_i x_j, \quad k = 1, 2, \\ x_3' &= x_3 = \sum_{j=1}^{3} x_3 x_j. \end{aligned} \tag{5}$$

For this operator, we have

$$x_2 = 1 - x_1 - x_3, \quad x_1 = x,$$

which yields the quadratic function

$$f(x) = Ax^2 + Bx + C. \tag{6}$$

From the equality

$$x_3' = x_1 x_3 + x_2 x_3 + x_3^2,$$

we obtain

$$P_{33,3} = 1, \quad P_{33,1} = P_{33,2} = 0,$$
$$P_{23,3} = \frac{1}{2}, \quad P_{13,3} = \frac{1}{2},$$

and

$$P_{12,1} + P_{12,2} + P_{12,3} = 1.$$

Let us introduce the following notations:

$$x_3 = R, \quad P_{11,1} = a_1, \quad P_{12,1} = a_2, \quad P_{13,1} = a_3,$$
$$P_{22,1} = b_1, \quad P_{23,1} = b_2, \quad P_{33,1} = 0.$$

Then the quadratic function takes the form

$$f(x) = (a_1 - 2a_2 + b_1)x^2 + 2\left((a_2 - b_1)(1 - R) + R(a_3 - b_2)\right)x + b_1(1 - R)^2 + 2b_2R(1 - R).$$

Thus,

$$A = a_1 - 2a_2 + b_1, \quad B = 2\left((a_2 - b_1)(1 - R) + R(a_3 - b_2)\right), \quad C = b_1(1 - R)^2 + 2b_2R(1 - R),$$

where

$$A \in [-2, 2] \setminus \{0\}, \quad B \in [-3, 3], \quad C \in [0, 1].$$

2|Fixed Points

We begin by determining the fixed points of the function

$$f(x) = Ax^2 + Bx + C.$$

Theorem 1. *If*

$$\begin{cases} C \le A, \\ B = 1 - 2\sqrt{AC}, \end{cases}$$

then the function $f(x)$ has a unique fixed point

$$x = \frac{1 - B}{2A} \in [0, 1].$$

Proof: Fixed points satisfy

$$Ax^2 + Bx + C = x,$$

which is equivalent to

$$Ax^2 + (B - 1)x + C = 0.$$

A unique solution exists when the discriminant is zero:

$$\Delta = (B - 1)^2 - 4AC = 0.$$

Hence,

$$B = 1 \pm 2\sqrt{AC}.$$

The case $B = 1 + 2\sqrt{AC}$ yields a negative solution, while $B = 1 - 2\sqrt{AC}$ gives

$$x = \frac{\sqrt{AC}}{A} \in [0, 1],$$

which holds if and only if $0 \le C \le A$. □

Theorem 2. *If the coefficients satisfy*

$$\begin{cases} -2 \le A < 0, \\ 2A + B < 1, \\ A + B + C \ge 1, \end{cases}$$

then $f(x)$ has a unique fixed point

$$x_1 = \frac{1 - B - \sqrt{(B - 1)^2 - 4AC}}{2A}.$$

Theorem 3. *Let $0 < A \le 2$. Then the function $f(x)$ has the fixed point*

$$x_1 = \frac{1 - B - \sqrt{(B - 1)^2 - 4AC}}{2A}$$

if

$$\begin{cases} 2A + B < 1, \\ A + B + C \le 1, \\ B > 1 - 2\sqrt{AC}, \end{cases} \quad \text{or} \quad \begin{cases} 2A + B > 1, \\ B > 1 - 2\sqrt{AC}. \end{cases}$$

Moreover, it has the fixed point

$$x_2 = \frac{1 - B + \sqrt{(B-1)^2 - 4AC}}{2A}$$

if

$$\begin{cases} B > 1 - 2\sqrt{AC}, \\ 2A + B > 1, \\ A + B + C \geq 1. \end{cases}$$

3|Type of Fixed Points

Theorem 4. (1) *If $\Delta = 0$, then the fixed point is a hyperbolic attracting fixed point.*

(2) *The fixed point x_1 is hyperbolic attracting when $\Delta < 4$, and hyperbolic repelling when $\Delta > 4$.*

(3) *The fixed point x_2 is a hyperbolic repelling fixed point.*

Proof: *i)* Consider the case $\Delta = 0$. In this case, we use topological conjugacy. Let

$$f(x) = Ax^2 + Bx + C, \quad g(x) = \mu x(1-x), \quad h(x) = mx + n.$$

We use the conjugacy relation

$$h \circ f = g \circ h.$$

Then

$$\begin{cases} h \circ f = mAx^2 + mBx + (mC + n), \\ g \circ h = -\mu m^2 x^2 + \mu m(1-2n)x + \mu(n - n^2). \end{cases}$$

Comparing coefficients, we obtain the system

$$\begin{cases} -\mu m = A, \\ \mu(1-2n) = B, \\ \mu(n - n^2) = mC + n. \end{cases}$$

Solving this system yields

$$\mu = 1, \quad m = -A, \quad n = \frac{1-B}{2}.$$

This implies that the fixed point is attracting.

ii) It is known that for the fixed point

$$x_1 = \frac{1 - B - \sqrt{(B-1)^2 - 4AC}}{2A},$$

The derivative satisfies

$$f'(x_1) = 2Ax_1 + B = 1 - \sqrt{\Delta}.$$

Therefore, the fixed point x_1 is hyperbolic attracting when $\Delta < 4$ and hyperbolic repelling when $\Delta > 4$.

iii) For the fixed point

$$x_2 = \frac{1 - B + \sqrt{(B-1)^2 - 4AC}}{2A},$$

we have

$$f'(x_2) = 2Ax_2 + B = 1 + \sqrt{\Delta}.$$

Hence, this fixed point is a hyperbolic repeller. □

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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