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# Certain Results on the Logarithmic Order and Logarithmic Type for Entire Matrix Functions in Hyperspherical Region

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**Abstract**

In this paper, we investigate the logarithmic growth of entire matrix functions of growth order zero. We show that the logarithmic order and the logarithmic type of entire functions in several complex matrix variables (FSCMVs) in hyperspherical regions. Examples are given to support the usability of our results. Finally, a result concerning linear substitution of the logarithmic growth of FSCMVs is given.

**Keywords:** Several complex matrices, Entire matrix functions, Hyperspherical region, Logarithmic order, Logarithmic type.

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# 1|Introduction and Preliminaries

Complex function theory focuses on analyzing the asymptotic growth behavior of complete functions in single and multiple complex variables. It covers a wide range of subjects related to the behavior of functions in one or several variables, including their relative growth. Distinctions such as growth order, type, maximal term, and central exponent can be used to compare the growth of these functions. Recent advances in this area have occurred in recent decades. For instance, consider [1, 2, 3, 4]. Further, several authors (see,e.g., [5, 6, 7, 8]) have explored generalizations to higher dimensions for the order and type of functions of multiple complex variables.

Moreover, Clifford analysis is used to generalize complex function theory to higher dimensions. This has led to research into the growth of whole monogenic functions (see [9, 10, 11]).

Recently, there has been a strong push to investigate matrix function theory. For in-depth applications of matrix function theory, see [12, 13]. Various problems affecting functions in several complex matrix variables have been approached from different perspectives in recent years, and several important conclusions have been obtained [14, 15]. In this regard, Kishka et al. [16] discovered the order and type of whole functions of two complex matrices in complete Reinhardt domains.

Later, Abul-Ez et al. [17] developed the concept of order and type of complete functions of numerous complex matrices in hyperspherical regions. For more information, check [17].

In this work, we introduce and study the logarithmic order and logarithmic type for entire functions of several complex matrices in hyperspherical regions and obtain their coefficient characterizations.

For classifying integral functions by their growth, the definition of order was introduced. If the order is a (finite) positive number, then the definition of type allows subclassification. For the classes of order $\rho = 0$ no subclassification is possible. For example, all integral functions that grow at least as fast as $e^{\ln z}$ have to be kept in one class. For this reason Iyer [18] introduced the concept of logarithmic order. Thus an entire function $f(z)$ of order zero, is said to be of logarithmic order $\rho^*$ if

$$\rho^* = \limsup_{r\to\infty} \frac{\ln\ln M(r)}{\ln\ln r}, \;\; 1 \le \rho^* \le \infty,$$

where

$$M(r) = M(r,f) = \max_{|z|=r} |f(z)|$$

is the maximum modulus of $f(z)$ in the closed disk $|z| \le r$. Analogously, lower logarithmic order $\lambda$ is defined as

$$\lambda = \liminf_{r\to\infty} \frac{\ln\ln M(r)}{\ln\ln r}, \;\; 1 \le \lambda \le \rho^* \le \infty.$$

For $1 < \rho^* < \infty$ the logarithmic type $T$ and lower logarithmic type $t$ are defined respectively it follows that

$$T = \limsup_{r\to\infty} \frac{\ln M(r)}{(\ln r)^{\rho^*}}, \quad 0 \le T \le \infty$$

and

$$t = \liminf_{r\to\infty} \frac{\ln M(r)}{(\ln r)^{\lambda}}, \;\; 0 \le t \le \infty.$$

Many articles involving logarithmic order, lower logarithmic order, logarithmic type, lower logarithmic type and the coefficient of Taylor series in one complex variable are obtained (cf. [5, 19, 20]).

Moreover, Metwally in [21] extended the following definitions of logarithmic order and logarithmic type of functions of single complex variable given by Iyer [18] and Rahman [5] to functions of two complex variables.

$$\mathcal{L}(\rho^*) = \limsup_{r\to\infty} \frac{\ln\ln M(r)}{\ln\ln r}, \quad 1 \le \mathcal{L}(\rho^*) \le \infty$$

and

$$\mathcal{L}(T) = \limsup_{r\to\infty} \frac{\ln M(r)}{(\ln r)^{\mathcal{L}(\rho^*)}},\ \ 0 \le \mathcal{L}(T) \le \infty,$$

where

$$M(r) = \max_{\overline{S}_r} \left|f(z_1; z_2)\right|.$$

He gave two results about the equivalent of the two theorems in Iyer [18] and Rahman [5] of each of the logarithmic order and logarithmic type of the functions of two complex variables $f(z_1; z_2)$ in the following forms

**Theorem 1.** *A necessary and sufficient condition that the entire matrix function* $f(z_1; z_2) = \sum_{m,n}^{\infty} a_{m,n} z_1^m\ z_2^n$ *to be of logarithmic order* $\mathcal{L}(\rho^*)$ *is that*

$$\begin{aligned}\mathcal{L}(\rho^*) =& \limsup_{r\to\infty}\ \frac{\ln\ln M(r)}{\ln\ln r}\\ =&1 + \limsup_{m+n\to\infty} \frac{\ln(m+n)}{\ln\left\{\ln\left(\left|\frac{\sigma_{m,n}}{a_{m,n}}\right|\right)^{\frac{1}{m+n}}\right\}}.\end{aligned} \tag{1}$$

**Theorem 2.** *If the entire matrix function* $f(z_1; z_2)$ *is of finite logarithmic order* $\mathcal{L}(\rho^*)$, $(1 < \mathcal{L}(\rho^*) < \infty)$. *Then the necessary and sufficient condition to be of logarithmic type* $\mathcal{L}(\tau)$ *is that*

$$\begin{aligned}\mathcal{L}(\tau) =& \limsup_{r\to\infty} \frac{\ln M(r)}{(\ln r)^{\mathcal{L}(\rho^*)}}\\ =&(\frac{1}{\mathcal{L}(\rho^*)})^{\mathcal{L}(\rho^*)}\ (\mathcal{L}(\rho^*) - 1)^{\mathcal{L}(\rho^*)-1}\ \limsup_{m+n\to\infty} \frac{(m+n)}{\left\{\ln\left(\left|\frac{\sigma_{m,n}}{a_{m,n}}\right|\right)^{\frac{1}{m+n}}\right\}^{\mathcal{L}(\rho^*)-1}}.\end{aligned} \tag{2}$$

Let $\mathbb{C}$ represent the field of complex numbers and $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$ the space of several complex numbers. An open spherical region of radius $r;\ \ r > 0$ is here denoted by $\mathbf{S}_r$ and its closure by $\overline{\mathbf{S}}_r$. In terms of the introduced notations, these region satisfy the inequalities (see, [16, 17])

$$\begin{aligned}\mathbf{S}_r &= \{\mathbf{z} \in \mathbb{C}^k : |\mathbf{z}| < r\},\\ \overline{\mathbf{S}}_r &= \{\mathbf{z} \in \mathbb{C}^k : |\mathbf{z}| \le r\}.\end{aligned} \tag{3}$$

Suppose that $\mathbf{X} = X_s = [x_{s;ij}];\ \ s = 1, 2, ..., n$ are commutative matrices in the complex space $\mathbb{C}^{N\times N}$ of complex matrices of common order $N$, where $x_{s;ij};\ \ i, j = 1, 2, ..., N$ are complex numbers, then the function $\mathcal{F}(\mathbf{X}) = [f_{s;ij}]_{1\le i,j<N}$ of several complex matrices is given by the following power series (cf. [17])

$$\begin{aligned}\mathcal{F}(\mathbf{X}) =& \sum_{(n_1,n_2,...,n_k)=\mathbf{0}}^{\infty} a_{n_1,n_2,...,n_k}\ X_1^{n_1}.X_2^{n_2}...X_k^{n_k}\\ =&\sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}}\mathbf{X}^{\mathbf{n}} = \sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}}\mathbf{Z}.\end{aligned} \tag{4}$$

Since $\mathbf{Z} = \mathbf{X}^{\mathbf{n}}$ and $\mathbf{Z} \in \mathcal{M}_N(\mathbb{C})$, thus we write

$$z_{s;ij} = \Pi_{s=1}^{k}\ X_s^{n_s} = X_1^{n_1}.X_2^{n_2}...X_k^{n_k}. \tag{5}$$

Hence

$$f_{s;ij} = \sum_{\mathbf{n}} a_{\mathbf{n}} z_{s;ij};\quad s = 1, 2, ..., k,\ \ i, j = 1, 2, ..., N. \tag{6}$$

Thus, we can say that the function $\mathcal{F}(\mathbf{X})$ will be convergent if the elements $f_{s;ij}$ given in (4) are convergent series for all $i$, $j = 1, 2, ..., N$. Consider the domain which is a subset of the space determined by the following inequalities

$$|\mathbf{X}| < \|R\,\mathbf{t}\| \;\Rightarrow\; |X_s| < \|R\,t_s\|;\; |\mathbf{t}| = 1, \quad s = 1, 2, ..., k, \tag{7}$$

where the symbol $|\mathbf{X}|$ denotes the matrices $(|x_{s;ij}|)$ whose elements are the modulii of the elements $x_{s;ij}$ of the matrices $\mathbf{X}$, and the symbol $\|a\|$ denotes a matrix each of its elements is equal to the positive number $a$.

Suppose that

$$\mathcal{F}(\mathbf{X}) = \sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} \mathbf{X}^{\mathbf{n}}$$

given above represent an entire function of several square complex matrices with Taylor expansion. See, [17] defined the maximum modulus of $\mathcal{F}(\mathbf{X})$ and Cauchy's inequality follows that

$$M[\mathcal{F}; \overline{\mathbf{S}}_R] = \max_{i,j} \max_{\overline{\mathbf{S}}_R} |\mathcal{F}(\mathbf{X})|, \tag{8}$$

$$|a_{\mathbf{n}}| \leq \frac{N M[\mathcal{F}; \overline{\mathbf{S}}_r]}{(rN)^{\sqsubset \mathbf{n} \sqsupset}} \sigma_{\mathbf{n}}, \tag{9}$$

where $\sqsubset \mathbf{n} \sqsupset = n_1 + n_2 + ... + n_k$. Since, the radius of regularity for the function of several complex matrices $\mathcal{F}(\mathbf{X})$ is infinity, i.e.,

$$\limsup_{\sqsubset \mathbf{n} \sqsupset \to \infty} \{N^{\sqsubset \mathbf{n} \sqsupset} |\frac{a_{\mathbf{n}}}{\sigma_{\mathbf{n}}}|\}^{\frac{1}{\sqsubset \mathbf{n} \sqsupset}} = 0. \tag{10}$$

**Definition 1.** (cf., [17]) Let $\mathcal{F}(\mathbf{X})$ be an entire function of several complex matrices. Then the order of growth of the maximum modulus of an entire function of several complex matrices is defined by

$$\rho = \limsup_{r \to \infty} \frac{\ln \ln \|M[\mathcal{F}; \overline{\mathbf{S}}_r]\|}{\ln r}. \tag{11}$$

The inferior order of growth is defined by

$$\delta = \liminf_{r \to \infty} \frac{\ln \ln \|M[\mathcal{F}; \overline{\mathbf{S}}_r]\|}{\ln r}. \tag{12}$$

**Definition 2.** (see, [17]) For any entire function of several complex matrices $\mathcal{F}(\mathbf{X})$ of order $\rho$ $(0 < \rho < \infty)$ the growth type $\tau$ is defined as

$$\tau = \limsup_{r \to \infty} \frac{\ln \|M[\mathcal{F}; \overline{\mathbf{S}}_r]\|}{r^{\rho}}. \tag{13}$$

For the entire function of several complex matrices $\mathcal{F}(\mathbf{X})$ given by a power matrix series in (4), [17] obtained the coefficient characterizations of $\rho$ and $\tau$.

# 2|Logarithmic order of FSCMVs

Let $\mathcal{F}(\mathbf{X})$ be an entire function of several complex matrix variables of common order $N$ and its growth order $\rho(\mathcal{F}) = 0$ with Taylor expansion

$$\mathcal{F}(\mathbf{X}) = \sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} \mathbf{X}^{\mathbf{n}} \tag{14}$$

and the maximum modulus

$$M[\mathcal{F}; \overline{\mathbf{S}}_R] = \max_{i,j} \max_{|X_1| < \|r\,t_1\| .. |X_k| < \|r\,t_k\|} |\mathcal{F}(\mathbf{X})|. \tag{15}$$

We shall prove the following lemma

**Lemma 1.** *If*

$$\limsup_{r\to\infty} \frac{\ln\ln \|M[\mathcal{F};\overline{\boldsymbol{S}}_r]\|}{\ln\ln r} \leq \gamma, \tag{16}$$

*then*

$$1+\limsup_{\sqsubset \boldsymbol{n}\sqsupset\to\infty} \frac{\ln(\sqsubset \boldsymbol{n}\sqsupset)}{\ln\left\{\ln\left(\frac{N\ \sigma_{\boldsymbol{n}}}{|a_{\boldsymbol{n}}|}\right)^{\frac{1}{\sqsubset \boldsymbol{n}\sqsupset}}\right\}} \leq \gamma. \tag{17}$$

*Proof*: If $\gamma=\infty$ then there is nothing to prove. therefore, at $\gamma_1>\gamma$ then for a suitable number $r_0$, (16)yields

$$\|M[\mathcal{F};\overline{\mathbf{S}}_r]\| < e^{(\ln r)^{\gamma_1}};\quad r>r_0.$$

From which and Cauchy's inequality in (9), we have

$$\begin{aligned}\left(\frac{|a_{\mathbf{n}}|}{N\ \sigma_{\mathbf{n}}}\right)^{\frac{1}{\sqsubset\mathbf{n}\sqsupset}} &\leq \min_{r>r_0} \frac{\exp(\frac{(\ln r)^{\gamma_1}}{\sqsubset\mathbf{n}\sqsupset})}{Nr}\\ &= \frac{\exp\left(\frac{1}{\sqsubset\mathbf{n}\sqsupset}(\frac{\sqsubset\mathbf{n}\sqsupset}{\gamma_1})^{\gamma_1/\gamma_1-1}\right)}{N\exp\left((\frac{\sqsubset\mathbf{n}\sqsupset}{\gamma_1})^{1/\gamma_1-1}\right)}.\end{aligned}$$

*Remark* 1. The previous minimum calculation

$$\begin{aligned} g =& \frac{\exp(\frac{(\ln r)^{\gamma_1}}{\sqsubset\mathbf{n}\sqsupset})}{Nr}\\ \ln g =& \frac{(\ln r)^{\gamma_1}}{\sqsubset \mathbf{n}\sqsupset} - \ln(Nr)\\ \frac{g'}{g} =& \frac{\gamma_1}{\sqsubset \mathbf{n}\sqsupset}\ \frac{(\ln r)^{\gamma_1-1}}{r} - \frac{1}{r},\end{aligned}$$

we get the minimum value of function when $g'=0$, then

$$\begin{aligned} 0 =& \frac{\gamma_1}{\sqsubset \mathbf{n}\sqsupset}\ \frac{(\ln r)^{\gamma_1-1}}{r} - \frac{1}{r}\\ \ln r =& (\frac{\sqsubset \mathbf{n}\sqsupset}{\gamma_1})^{\frac{1}{\gamma_1-1}}\\ r =& \exp\left((\frac{\sqsubset \mathbf{n}\sqsupset}{\gamma_1})^{\frac{1}{\gamma_1-1}}\right).\end{aligned}$$

Thus

$$\begin{aligned}\ln\left(\frac{N\ \sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset\mathbf{n}\sqsupset}} \geq& \ln\left(\frac{N\exp\left((\frac{\sqsubset\mathbf{n}\sqsupset}{\gamma_1})^{1/\gamma_1-1}\right)}{\exp\left(\frac{1}{\gamma_1}(\frac{\sqsubset\mathbf{n}\sqsupset}{1})^{\gamma_1/\gamma_1-1}\right)}\right)\\ \geq& \ln N + (\frac{\sqsubset \mathbf{n}\sqsupset}{\gamma_1})^{\frac{1}{\gamma_1-1}} - \frac{1}{\gamma_1}(\frac{\sqsubset \mathbf{n}\sqsupset}{\gamma_1})^{\frac{1}{\gamma_1-1}}\\ =& (\frac{\sqsubset \mathbf{n}\sqsupset}{\gamma_1})^{\frac{1}{\gamma_1-1}}\Big[1-\frac{1}{\gamma_1}+(\frac{\gamma_1}{\sqsubset \mathbf{n}\sqsupset})^{\frac{1}{\gamma_1-1}}\ \ln N\Big].\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&1+\limsup_{\sqsubset \mathbf{n} \sqsupset \to\infty} \frac{\ln(\sqsubset \mathbf{n} \sqsupset)}{\ln\left\{\ln\left(\frac{N\,\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset \mathbf{n} \sqsupset}}\right\}} \\
&\leq\ 1+\limsup_{\sqsubset \mathbf{n} \sqsupset \to\infty} \frac{(\gamma_1-1)\ln(\sqsubset \mathbf{n} \sqsupset)}{\ln(\sqsubset \mathbf{n} \sqsupset)-\ln\gamma_1+(\gamma_1-1)\ln\left[1-\frac{1}{\gamma_1}+(\frac{\gamma_1}{\sqsubset \mathbf{n} \sqsupset})^{\frac{1}{\gamma_1-1}}\ \ln N\right]}\ \leq\ \gamma_1.
\end{aligned}
\tag{18}
$$

Since $\gamma_1$ can be chosen arbitrarily close to $\gamma$, we thus obtain the result (17) and the Lemma 1 is proved. □

One the other hand, we prove the following Lemma:

**Lemma 2.** *Suppose that*

$$
1+\limsup_{\sqsubset \boldsymbol{n} \sqsupset \to\infty} \frac{\ln(\sqsubset \boldsymbol{n} \sqsupset)}{\ln\left\{\ln\left(\frac{N\,\sigma_{\boldsymbol{n}}}{|a_{\boldsymbol{n}}|}\right)^{\frac{1}{\sqsubset \boldsymbol{n} \sqsupset}}\right\}} \leq \gamma. \tag{19}
$$

*Then*

$$
\limsup_{r\to\infty} \frac{\ln\ln\|M[\mathcal{F};\overline{\boldsymbol{S}}_r]\|}{\ln\ln r} \leq \gamma. \tag{20}
$$

*Proof*: Also, if $\gamma=\infty$ then there is nothing to prove. Thus if $\gamma<\infty$, choose any finite number $\gamma_1>\gamma-1$, in view of (19) there exists an integer $\mu$ such that

$$
\frac{\ln(\sqsubset \mathbf{n} \sqsupset)}{\ln\left\{\ln\left(\frac{N\,\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset \mathbf{n} \sqsupset}}\right\}} \leq \gamma_1, \tag{21}
$$

that is

$$
\begin{aligned}
\ln\ln\left(\frac{N\,\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset \mathbf{n} \sqsupset}} &\geq \frac{1}{\gamma_1}\ln \sqsubset \mathbf{n} \sqsupset \\
\left(\frac{N\,\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset \mathbf{n} \sqsupset}} &\geq \exp(\sqsubset \mathbf{n} \sqsupset)^{\frac{1}{\gamma_1}} \\
\left(\frac{|a_{\mathbf{n}}|}{N\,\sigma_{\mathbf{n}}}\right) &\leq \exp(-\sqsubset \mathbf{n} \sqsupset)^{\frac{1}{\gamma_1}+1}
\end{aligned}
\tag{22}
$$

Thus

$$
\begin{aligned}
\|\ M[\mathcal{F};\overline{\mathbf{S}}_r]\ \| &\leq \max_{i,j}\max_{\overline{\mathbf{S}}_r}\|\sum_{\mathbf{n}} a_{\mathbf{n}}X^{\mathbf{n}}\| \\
&\leq \frac{1}{N}\sum_{\mathbf{n}=\mathbf{0}}^{\infty}(Nr)^{\sqsubset \mathbf{n} \sqsupset}\frac{|a_{\mathbf{n}}|}{\sigma_{\mathbf{n}}} \\
&\leq \sum_{\mathbf{n}=\mathbf{0}}^{\infty}(Nr)^{\sqsubset \mathbf{n} \sqsupset}\exp(-\sqsubset \mathbf{n} \sqsupset)^{\frac{1}{\gamma_1}+1}.
\end{aligned}
\tag{23}
$$

Inserting (22) in (23) it follows that

$$
\begin{aligned}
\|\ M[\mathcal{F};\overline{\mathbf{S}}_r]\ \| &\leq \mathcal{C}_1+\sum_{\mathbf{n}>\mu}\left(\frac{Nr}{\exp\left((\sqsubset \mathbf{n} \sqsupset)^{\frac{1}{\gamma_1}}\right)}\right)^{\sqsubset \mathbf{n} \sqsupset} \\
&< \mathcal{C}_1+\sum_{\nu>\mu}(\nu+1)\Big(\frac{Nr}{\exp\left((\nu)^{\frac{1}{\gamma_1}}\right)}\Big)^{\nu},
\end{aligned}
$$

where $\mathcal{C}_1=\sum_{\mathbf{n}\leq\mu}(Nr)^{\sqsubset \mathbf{n} \sqsupset}\frac{|a_{\mathbf{n}}|}{\sigma_{\mathbf{n}}}$. Now, there is a number $r_0>1$ such that

$$(1+\ln r)^{\gamma_1} > \mu, \text{ for } r > r_0.$$

Then the integer $\varepsilon$ can be fixed such that

$$\mu < \varepsilon \leq (1+\ln r)^{\gamma_1} < \varepsilon + 1; \; r > r_0. \tag{24}$$

We consider the following sums

$$\begin{aligned}\sum_{\nu=\mu+1}^{\varepsilon} (\nu+1)\Big(\frac{Nr}{\exp\Big((\nu)^{\frac{1}{\gamma_1}}\Big)}\Big)^{\nu} &< (Nr)^{\varepsilon} \sum_{\nu=\mu+1}^{\varepsilon} (\nu+1)\Big(\frac{Nr}{\exp\Big((\mu)^{\frac{1}{\gamma_1}}\Big)}\Big)^{\nu} \\ &< (Nr)^{\varepsilon} \sum_{\nu=0}^{\infty} (\nu+1)\Big(\frac{Nr}{\exp\Big((\mu)^{\frac{1}{\gamma_1}}\Big)}\Big)^{\nu} \\ &< (Nr)^{(1+\ln r)^{\gamma_1}} (1-\exp\Big((-\mu)^{\frac{1}{\gamma_1}}\Big)^{-2} < \mathcal{C}_2\,(Nr)^{(1+\ln r)^{\gamma_1}},\end{aligned} \tag{25}$$

and

$$\sum_{\nu=\varepsilon+1}^{\infty} (\nu+1)\Big(\frac{Nr}{\exp\Big((\nu)^{\frac{1}{\gamma_1}}\Big)}\Big)^{\nu} < \sum_{\nu=\varepsilon+1}^{\infty} (\nu+1)/e^{\nu} = (1-1/e)^{-2} < \mathcal{C}_3, \tag{26}$$

where $\mathcal{C}_2$ and $\mathcal{C}_3$ are a finite constants. Therefore, from (22), (25) and (26), we have

$$\begin{aligned}\| M[\mathcal{F};\overline{\mathbf{S}}_r] \| &< \mathcal{C}_1 + \mathcal{C}_3 + \mathcal{C}_2\; (Nr)^{(1+\ln r)^{\gamma_1}} \\ &< (Nr)^{(1+\ln r)^{\gamma_1}} \Big(\mathcal{C}_2 + \frac{\mathcal{C}_1+\mathcal{C}_3}{(Nr)^{(1+\ln r)^{\gamma_1}}}\Big).\end{aligned} \tag{27}$$

Making $r$ tends to infinity, we infer that

$$\limsup_{r\to\infty} \frac{\ln\ln \|M[\mathcal{F};\overline{\mathbf{S}}_r]\|}{\ln\ln r} \leq \gamma_1 + 1. \tag{28}$$

Since $\gamma_1$ is arbitrarily chosen then inequality (20) follows. □

Combining Lemmas 1 and 2, we obtain the generalization of Theorem 1 due to Metwally (cf. [21]) concerning the logarithmic order of the entire matrix function $\mathcal{F}(\mathbf{X})$.

**Theorem 3.** *A necessary and sufficient condition that the entire matrix function* $\mathcal{F}(\boldsymbol{X}) = \sum_{\boldsymbol{n}=\boldsymbol{0}}^{\infty} a_{\boldsymbol{n}}\,\boldsymbol{X}^{\boldsymbol{n}}$ *to be of logarithmic order* $\mathcal{L}(\rho)$ *is that*

$$\mathcal{L}(\rho) = 1 + \limsup_{\sqsubset \boldsymbol{n} \sqsupset\to\infty} \frac{\ln(\sqsubset \boldsymbol{n} \sqsupset)}{\ln\left\{\ln\left(\frac{N\,\sigma_{\boldsymbol{n}}}{|a_{\boldsymbol{n}}|}\right)^{\frac{1}{\sqsubset \boldsymbol{n} \sqsupset}}\right\}}. \tag{29}$$

*Remark* 2. The procedure adopted here in Lammas 1 and 2 is similar to that followed in two complex variable case (cf. [21])

# 3|Logarithmic order of FSCMVs

Now, we introduce another result concerning the logarithmic type of the entire matrix function $\mathcal{F}(\mathbf{X})$, this result is formulated in the following:

**Lemma 3.** *If*

$$\limsup_{r\to\infty} \frac{\ln\|M[\mathcal{F};\overline{\boldsymbol{S}}_r]\|}{(\ln r)^{\mathcal{L}(\rho)}} \leq \; \gamma, \tag{30}$$

*then*

$$(\frac{1}{\mathcal{L}(\rho)})^{\mathcal{L}(\rho)}\ (\mathcal{L}(\rho)-1)^{\mathcal{L}(\rho)-1}\ \limsup_{\sqsubset \boldsymbol{n}\sqsupset\to\infty} \frac{(\sqsubset \boldsymbol{n} \sqsupset)}{\left\{\ln\left(\frac{N\sigma_{\boldsymbol{n}}}{|a_{\boldsymbol{n}}|}\right)^{\frac{1}{\sqsubset \boldsymbol{n}\sqsupset}}\right\}^{\mathcal{L}(\rho)-1}} \leq\ \gamma, \tag{31}$$

*Proof*: Let $\gamma_1$ be any finite number greater than $\gamma$, then from (30) there is a number $r_0$ such that

$$\|M[\mathcal{F};\overline{\mathbf{S}}_r]\| < \exp(\gamma_1(\ln r)^{\mathcal{L}(\rho)}), \quad r > r_0. \tag{32}$$

Using Cauchy's inequality we find that

$$\begin{aligned}\left(\frac{|a_{\mathbf{n}}|}{N\ \sigma_{\mathbf{n}}}\right)^{\frac{1}{\sqsubset\mathbf{n}\sqsupset}} &\leq \min_{r>r_0} \frac{\exp(\frac{(\ln r)^{\mathcal{L}(\rho)}\gamma_1}{\sqsubset\mathbf{n}\sqsupset})}{Nr}\\ &= \frac{\exp\left(\frac{\gamma_1}{\sqsubset\mathbf{n}\sqsupset}(\frac{\sqsubset\mathbf{n}\sqsupset}{\mathcal{L}(\rho)\gamma_1})^{(\mathcal{L}(\rho)/\mathcal{L}(\rho)-1)}\right)}{N\exp\left((\frac{\sqsubset\mathbf{n}\sqsupset}{\gamma_1\mathcal{L}(\rho)})^{1/\mathcal{L}(\rho)-1}\right)}.\end{aligned}$$

*Remark* 3. The previous minimum calculation

$$\begin{aligned}h =&\frac{\exp(\frac{(\ln r)^{\mathcal{L}(\rho)}\gamma_1}{\sqsubset\mathbf{n}\sqsupset})}{Nr}\\ \ln h =&\frac{(\ln r)^{\mathcal{L}(\rho)}\gamma_1}{\sqsubset \mathbf{n} \sqsupset} - \ln(Nr)\\ \frac{h'}{h} =&\frac{\gamma_1}{\sqsubset \mathbf{n} \sqsupset}\ \mathcal{L}(\rho)(\ln r)^{\mathcal{L}(\rho)-1}\ \frac{1}{r} - \frac{1}{r},\end{aligned}$$

we get the minimum value of function when $h' = 0$, then

$$\begin{aligned}0 =&\frac{\gamma_1}{\sqsubset \mathbf{n} \sqsupset}\ \mathcal{L}(\rho)(\ln r)^{\mathcal{L}(\rho)-1}\ \frac{1}{r} - \frac{1}{r}\\ r =&\exp\left((\frac{\sqsubset \mathbf{n} \sqsupset}{\gamma_1\mathcal{L}(\rho)})^{1/\mathcal{L}(\rho)-1}\right).\end{aligned}$$

Hence, we have

$$\limsup_{\sqsubset\mathbf{n}\sqsupset\to\infty} \frac{(\sqsubset \mathbf{n} \sqsupset)}{\left\{\ln\left(\frac{N\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset\mathbf{n}\sqsupset}}\right\}^{\mathcal{L}(\rho)-1}} \leq\ \gamma_1\ \mathcal{L}(\rho)(\frac{\mathcal{L}(\rho)}{\mathcal{L}(\rho)-1})^{\mathcal{L}(\rho)-1}. \tag{33}$$

Since $\gamma_1$ can be taken close enough to $\gamma$, then

$$(\frac{1}{\mathcal{L}(\rho)})^{\mathcal{L}(\rho)}\ (\mathcal{L}(\rho)-1)^{\mathcal{L}(\rho)-1}\ \limsup_{\sqsubset\mathbf{n}\sqsupset\to\infty} \frac{(\sqsubset \mathbf{n} \sqsupset)}{\left\{\ln\left(\frac{N\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset\mathbf{n}\sqsupset}}\right\}^{\mathcal{L}(\rho)-1}} \leq\ \gamma, \tag{34}$$

and the Lemma 3 is proved. □

Also, we establish the following

**Lemma 4.** *Suppose that*

$$(\frac{1}{\mathcal{L}(\rho)})^{\mathcal{L}(\rho)}\ (\mathcal{L}(\rho)-1)^{\mathcal{L}(\rho)-1}\ \limsup_{\sqsubset \boldsymbol{n} \sqsupset\to\infty} \frac{(\sqsubset \boldsymbol{n} \sqsupset)}{\left\{\ln\left(\frac{N\sigma_{\boldsymbol{n}}}{|a_{\boldsymbol{n}}|}\right)^{\frac{1}{\sqsubset \boldsymbol{n}\sqsupset}}\right\}^{\mathcal{L}(\rho)-1}} \leq\ \gamma, \tag{35}$$

*then*

$$\limsup_{r\to\infty} \frac{\ln \|M[\mathcal{F}; \overline{\boldsymbol{S}}_r]\|}{(\ln r)^{\mathcal{L}(\rho)}} \leq\ \gamma, \tag{36}$$

*Proof*: If $\gamma$ is finite choose the finite number $\gamma_1 > \gamma$ and $\mu > 1$, then from (35) we see that

$$\frac{(\sqsubset \mathbf{n} \sqsupset)}{\left\{\ln\left(\frac{N\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset \mathbf{n}\sqsupset}}\right\}^{\mathcal{L}(\rho)-1}} \leq \gamma_1\ \mathcal{L}(\rho)\Big(\frac{\mathcal{L}(\rho)-1}{\mathcal{L}(\rho)}\Big)^{1-\mathcal{L}(\rho)}$$
$$\frac{1}{\left\{\ln\left(\frac{N\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\right)^{\frac{1}{\sqsubset \mathbf{n}\sqsupset}}\right\}^{\mathcal{L}(\rho)-1}} \leq \frac{\gamma_1\ \mathcal{L}(\rho)}{\sqsubset \mathbf{n} \sqsupset}\Big(\frac{\mathcal{L}(\rho)-1}{\mathcal{L}(\rho)}\Big)^{1-\mathcal{L}(\rho)},$$

thus,

$$\ln\Big(\frac{N\sigma_{\mathbf{n}}}{|a_{\mathbf{n}}|}\Big)^{\frac{1}{\sqsubset \mathbf{n}\sqsupset}} \geq \Big(\frac{\sqsubset \mathbf{n} \sqsupset}{\gamma_1 \mathcal{L}(\rho)}\Big)^{\frac{1}{\mathcal{L}(\rho)-1}}\ \Big(\frac{\mathcal{L}(\rho)-1}{\mathcal{L}(\rho)}\Big)$$
$$\frac{|a_{\mathbf{n}}|}{N\ \sigma_{\mathbf{n}}}\ \leq\ \exp\Bigg(-(\sqsubset \mathbf{n} \sqsupset)(\frac{\mathcal{L}(\rho)-1}{\mathcal{L}(\rho)})(\frac{\sqsubset \mathbf{n} \sqsupset}{\gamma_1 \mathcal{L}(\rho)})^{\frac{1}{(\mathcal{L}(\rho)-1)}}\Bigg); \quad \sqsubset \mathbf{n} \sqsupset > \mu. \tag{37}$$

From which we have

$$\| M[\mathcal{F}; \overline{\mathbf{S}}_r] \| \leq \frac{1}{N}\sum_{\mathbf{n}=\mathbf{0}}^{\infty}(Nr)^{\sqsubset \mathbf{n} \sqsupset}\frac{|a_{\mathbf{n}}|}{\sigma_{\mathbf{n}}}$$
$$< A_1 + \sum_{\sqsubset \mathbf{n} \sqsupset > \mu} \exp\Bigg(-(\sqsubset \mathbf{n} \sqsupset)(\frac{\mathcal{L}(\rho)-1}{\mathcal{L}(\rho)})\ (\frac{\sqsubset \mathbf{n} \sqsupset}{\gamma_1 \mathcal{L}(\rho)})^{1/(\mathcal{L}(\rho)-1)}\Bigg)\ (Nr)^{\sqsubset \mathbf{n} \sqsupset} \tag{38}$$

Choose the number $r_0 > 1$ and fixed the integer $\epsilon$ such that

$$\gamma_1\ \mathcal{L}(\rho)(1+\ln r)^{\mathcal{L}(\rho)-1} > \mu + 1,$$

and

$$\epsilon \leq (1+\ln r)^{\mathcal{L}(\rho)-1}\ \mathcal{L}(\rho)\ \gamma_1\ (\frac{\mathcal{L}(\rho)}{\mathcal{L}(\rho)-1})^{\mathcal{L}(\rho)-1} < \epsilon + 1; \quad r > r_0. \tag{39}$$

By which and relation (38), we get

$$\begin{aligned}\parallel M[\mathcal{F};\overline{\mathbf{S}}_r]\parallel &< A_1+\sum_{\nu=\mu+1}^{\epsilon}(\nu+1)\left\{\frac{r^{\mathcal{L}(\rho)}}{\exp\left(\mathcal{L}(\rho)-1(\frac{}{\gamma_1\mathcal{L}(\rho)})^{1/(\mathcal{L}(\rho)-1)}\right)}\right\}^{\nu/\mathcal{L}(\rho)}\\ &+\sum_{\nu=\epsilon+1}^{\infty}(\nu+1)\left\{\frac{r^{\mathcal{L}(\rho)}}{\exp\left(\mathcal{L}(\rho)-1(\frac{}{\gamma_1\mathcal{L}(\rho)})^{1/(\mathcal{L}(\rho)-1)}\right)}\right\}^{\nu/\mathcal{L}(\rho)} \\ &< A_1+\sum_{\nu=\mu+1}^{\epsilon}(\nu+1)\; r^{\gamma_1(\ln r)^{\mathcal{L}(\rho)-1}}+\sum_{\nu=\epsilon+1}^{\infty}\frac{(\nu+1)}{e^{\nu}}\\ &< A_1+\frac{1}{(1-1/e)^{-2}}+\epsilon\;(\epsilon+1)\; r^{\gamma_1\;(\ln r)^{\mathcal{L}(\rho)-1}}.\end{aligned} \tag{40}$$

Therefore, we obtain

$$\limsup_{r\to\infty}\frac{ln\|M[\mathcal{F};\overline{\mathbf{S}}_r]\|}{(lnr)^{\mathcal{L}(\rho)}}\leq\;\gamma_1. \tag{41}$$

Keeping in mind that $\gamma_1$ is arbitrarily chosen we infer that the relation (36). □

Combining these two Lemmas, we obtain the coefficient characterization, for logarithmic type, for entire functions of several complex matrices.

Therefore, the following theorem provides us with a generalization of the theorem of Metwally on the relation between the type and the Taylor coefficients (cf. [21]) to the matrix analysis setting.

**Theorem 4.** *If the entire matrix function $\mathcal{F}(\boldsymbol{X})$ is of finite logarithmic order $\mathcal{L}(\rho)$, $(1<\rho<\infty)$. Then the necessary and sufficient condition to be of logarithmic type $\mathcal{L}(\tau)$ is that*

$$\mathcal{L}(\tau)=(\frac{1}{\mathcal{L}(\rho)})^{\mathcal{L}(\rho)}\;(\mathcal{L}(\rho)-1)^{\mathcal{L}(\rho)-1}\;\limsup_{\sqsubset\boldsymbol{n}\sqsupset\to\infty}\frac{(\sqsubset\boldsymbol{n}\sqsupset)}{\left\{\ln\left(\frac{\sigma_{\boldsymbol{n}}}{|a_{\boldsymbol{n}}|N^{\sqsubset\boldsymbol{n}\sqsupset}}\right)^{\frac{1}{\sqsubset\boldsymbol{n}\sqsupset}}\right\}^{\mathcal{L}(\rho)-1}}. \tag{42}$$

Now, we will give some examples for Theorem 3 and Theorem 4.

**Example 1.** *According to* [17] *the growth order of the matrix function*

$$\mathcal{F}(\boldsymbol{X})=\sum_{\boldsymbol{n}=1}^{\infty}\left(\frac{1}{\sqsubset\boldsymbol{n}\sqsupset}\right)^{\sqsubset\boldsymbol{n}\sqsupset^2}\boldsymbol{X}^{\boldsymbol{n}}$$

*is zero,*

$$\begin{aligned}\rho&=\limsup_{\sqsubset\boldsymbol{n}\sqsupset\to\infty}\frac{\sqsubset\boldsymbol{n}\sqsupset\ln(\sqsubset\boldsymbol{n}\sqsupset)}{-\ln\left|\frac{a_{\boldsymbol{n}}}{K_{\boldsymbol{n}}}\right|},\quad K_{\boldsymbol{n}}=\frac{\sigma_{\boldsymbol{n}}}{N^{\sqsubset\boldsymbol{n}\sqsupset}}\\ &=\limsup_{\sqsubset\boldsymbol{n}\sqsupset\to\infty}\frac{\sqsubset\boldsymbol{n}\sqsupset\ln(\sqsubset\boldsymbol{n}\sqsupset)}{-\ln\left|\frac{\left(\frac{1}{\sqsubset\boldsymbol{n}\sqsupset}\right)^{\sqsubset\boldsymbol{n}\sqsupset^2}}{K_{\boldsymbol{n}}}\right|}\\ &=\limsup_{\sqsubset\boldsymbol{n}\sqsupset\to\infty}\frac{\sqsubset\boldsymbol{n}\sqsupset\ln(\sqsubset\boldsymbol{n}\sqsupset)}{\ln\sigma_{\boldsymbol{n}}+\sqsubset\boldsymbol{n}\sqsupset^2\;\ln\sqsubset\boldsymbol{n}\sqsupset-\sqsubset\boldsymbol{n}\sqsupset\;\ln N}=0.\end{aligned}$$

*Therefore, applying Theorem 3 and Theorem 4 lead immediately to*

$$\begin{aligned}\mathcal{L}(\rho) =&1+\limsup_{\sqsubset \boldsymbol{n} \sqsupset\to\infty}\frac{\ln(\sqsubset \boldsymbol{n} \sqsupset)}{\ln\left\{\ln\left(\frac{N\ \sigma_n}{|a_n|}\right)^{\frac{1}{\sqsubset n \sqsupset}}\right\}}\\ =&1+\limsup_{\sqsubset \boldsymbol{n} \sqsupset\to\infty}\frac{\ln(\sqsubset \boldsymbol{n} \sqsupset)}{\ln(\sqsubset \boldsymbol{n} \sqsupset)+\ln\ln(\sqsubset \boldsymbol{n} \sqsupset)+\ln\left\{1+\frac{\ln(N\ \sigma_n)}{\sqsubset \boldsymbol{n} \sqsupset^2\ln(\sqsubset \boldsymbol{n} \sqsupset)}\right\}}=2\end{aligned}$$

*and*

$$\begin{aligned}\mathcal{L}(\tau) =&(\frac{1}{\mathcal{L}(\rho)})^{\mathcal{L}(\rho)}\ (\mathcal{L}(\rho)-1)^{\mathcal{L}(\rho)-1}\ \limsup_{\sqsubset \boldsymbol{n} \sqsupset\to\infty}\frac{(\sqsubset \boldsymbol{n} \sqsupset)}{\left\{\ln\left(\frac{\sigma_n}{|a_n|N^{\sqsubset n \sqsupset}}\right)^{\frac{1}{\sqsubset n \sqsupset}}\right\}^{\mathcal{L}(\rho)-1}}\\ =&(\frac{1}{2})^2\ \limsup_{\sqsubset \boldsymbol{n} \sqsupset\to\infty}\frac{(\sqsubset \boldsymbol{n} \sqsupset)}{(\sqsubset \boldsymbol{n} \sqsupset)\ln(\sqsubset \boldsymbol{n} \sqsupset)+\frac{1}{\sqsubset \boldsymbol{n} \sqsupset}\ \ln\sigma_{\boldsymbol{n}}-\ln N}=0.\end{aligned}$$

*Thus, the given matrix function is of logarithmic order $\mathcal{L}(\rho)=2$ and logarithmic type $\mathcal{L}(\tau)=0$.*

**Example 2.** *Let*

$$\mathcal{F}(\boldsymbol{X}):=e^{\sum_{s=1}^{k}(\ln X_s)^{\alpha}};\ s=1,2,...,k,\ \alpha>0,$$

*be an entire function of the several square complex matrices $\boldsymbol{X}=(X_1,X_2,...,X_k)$ of which are of common order $N$ in $\overline{\boldsymbol{S}}_r$; $b_s$ are positive numbers, then the growth order $\rho=0$ according to* [15]*. Since*

$$M[\mathcal{F};\overline{\boldsymbol{S}}_r]=e^{k(\ln r)^{\alpha}},$$

*applying the relations* (15) *and* (27)*, we get*

$$\mathcal{L}(\rho)=\limsup_{r\to\infty}\frac{\ln\ln\|M[\mathcal{F};\overline{\boldsymbol{S}}_r]\|}{\ln\ln r}=\alpha,$$

$$\mathcal{L}(\tau)=\limsup_{r\to\infty}\frac{ln\|M[\mathcal{F};\overline{\boldsymbol{S}}_r]\|}{(lnr)^{\mathcal{L}(\rho)}}=k.$$

# 4|Linear substitution for FSCMVs

Let $\mathcal{F}^{\star}(\mathbf{X})=\mathcal{F}(\mathbf{X}+\mathbf{A})$ be an entire matrix function of logarithmic order $\mathcal{L}(\rho)$ and logarithmic type $\mathcal{L}(\tau)$, where $\mathbf{A}=[a_{s;ij}];\ s=1,2,...,k;\ i,j=1,2,...,N$ are any constant numbers. It is required in this section to establish $\mathcal{L}(\rho^{\star})$ and $\mathcal{L}(\tau^{\star})$ as follows:

If $\mathbf{X}=([x_{s;ij}]);\ s=1,2,...,k$ are several complex matrices situated in the neighbourhood of the origin by (15), then

$$\begin{aligned}\sum_{s=1}^{k}(|x_{s;ij}+a_{s;ij}|)^s\leq&\sup_{|\mathbf{t}|=1}\Big(r^k+2r(t_1+t_2+...+t_k)+...+k\theta^k\Big)\\ =&(r+\sqrt{k}\ \theta)^k=R^k;\ \ \theta=\sup_{i,j}(|a_{1;ij}|,|a_{2;ij}|,...,|a_{k;ij}|).\end{aligned}$$

So that the matrices $\mathbf{X+A}$ are situated in the neighbourhood of the origin given by

$$|\mathbf{X+A}|<\|R\ \mathbf{t}\|;\ \ |\mathbf{t}|=1,\ \ R=r+\sqrt{k}\ \theta. \tag{43}$$

Thus

$$M[\mathcal{F}^{\star};\overline{\mathbf{S}}_r]\leq M[\mathcal{F};\overline{\mathbf{S}}_r]. \tag{44}$$

Now, suppose that $\epsilon > 0$ is any number, then there is a positive number $r_1$ such that, applying Theorem 3 and (44) one gets

$$M[\mathcal{F}^\star;\overline{\mathbf{S}}_r] \leq M[\mathcal{F};\overline{\mathbf{S}}_r] < e^{\epsilon \ln R^{\mathcal{L}(\rho)}} \quad \text{for } R > r_1. \tag{45}$$

That is

$$\begin{aligned}\mathcal{L}(\rho^\star) = & \quad \limsup_{r\to\infty} \tfrac{\ln\ln M[\mathcal{F}^\star;\overline{\mathbf{S}}_r]}{\ln\ln r} \\ \leq & \quad \limsup_{r\to\infty} \tfrac{\ln \epsilon(r+\sqrt{k}\ \theta)^{\mathcal{L}(\rho)}}{\ln\ln r} = \mathcal{L}(\rho).\end{aligned}$$

On the other hand, if $\mathcal{F}^\star(\mathbf{X}) = \mathcal{F}(\mathbf{X}\text{-}\mathbf{A})$, then $\mathcal{L}(\rho) \leq \mathcal{L}(\rho^\star)$ and thus $\mathcal{L}(\rho) = \mathcal{L}(\rho^\star)$.

Therefore, the type $\tau^\star$, it follows from (42) that

$$\begin{aligned}\mathcal{L}(\tau^\star) =& \limsup_{r\to\infty} \frac{\ln M[\mathcal{F}^\star;\overline{\mathbf{S}}_r]}{r^{\mathcal{L}(\rho^\star)}} \\ \leq& \limsup_{r\to\infty} \frac{\epsilon(r+\sqrt{k}\ \theta)^{\mathcal{L}(\rho)}}{r^{\mathcal{L}(\rho)}} = \epsilon.\end{aligned} \tag{46}$$

Since $\epsilon$ can be chosen arbitrarily as near to $\mathcal{L}(\tau)$ as we please, we infer that $\mathcal{L}(\tau^\star) \leq \mathcal{L}(\tau)$. In the usual way, we infer that $\mathcal{L}(\tau^\star) \geq \mathcal{L}(\tau)$ and hence $\mathcal{L}(\tau^\star) = \mathcal{L}(\tau)$. Thus, the following theorem is thus determined

**Theorem 5.** *The entire matrix function $\mathcal{F}^\star(\boldsymbol{X})$ is of the same logarithmic order and logarithmic type as the matrix function $\mathcal{F}(\boldsymbol{X})$.*

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