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Links between Arithmetic Functions and Integer Partitions

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Abstract

This article explores the deep connections between arithmetic functions and integer partitions through the application of the Fine and Jameson-Schneider theorems. By employing Bell polynomials, several classical arithmetic functions, including the divisor function, sum of divisors, Euler's totient function, Möbius function, and sums of two squares, are represented in terms of integer partitions. The study highlights how combinatorial structures provide alternative approaches to evaluate and interpret number-theoretic functions. Furthermore, recurrence relations and identities are established, enriching the theoretical framework linking partition theory with analytic number theory. These results contribute to a broader understanding of arithmetic properties and their combinatorial representations, offering potential applications in both pure mathematics and related computational fields.

Keywords: Divisor function, Jameson-Schneider theorem, Integer partitions, Bell polynomials, Fine's theorem, Arithmetic functions, Sums of two squares, Euler's totient function, Nontrivial Dirichlet character (mod 4), Möbius function, Jacobi's identity.

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1|Introduction

Here we consider the functions:

$$\psi_j(q) := (1-q)^{s(j)} = \sum_{n=0}^{\infty} C_j(n) q^n, \tag{1}$$

for any sequence $\{s(n)\}$, and we accept that:

$$\eta(q) := \Pi_{j=1}^{\infty}\ \psi_j(q^j) = \Pi_{j=1}^{\infty}\ (1-q^j)^{s(j)} = \sum_{n=0}^{\infty} R(n) q^n, \quad R(0)=1; \tag{2}$$

on the other hand, Jameson-Schneider [1, 2] proved the following result:

$$q\ \frac{d}{dq}\ ln\ \eta = -\sum_{n=1}^{\infty}\Big(\sum_{d/n} s(d)\ d\Big) q^n. \tag{3}$$

Besides, we know the Fine's theorem [3, 4, 5]:

$$\eta = \sum_{n=0}^{\infty}\Big(\sum_{\lambda\ \vdash\ n} C_1(k_1) C_2(k_2)\ ...\ C_n(k_n)\Big) q^n, \tag{4}$$

such that $\lambda\ \vdash\ n$ means all partitions of n, and k_r is the multiplicity of r in a given partition; therefore, (2) and (4) imply the connection:

$$R(n) = \sum_{\lambda \vdash n} C_1(k_1) C_2(k_2)\ ...\ C_n(k_n) \tag{5}$$

In Sec. 2 we apply the expressions (1), ... , (5) to several sequences $\{s(n)\}$, in particular, the case $s(j) = 1/j$ shows that the partitions of an integer can give the number of divisors of it.

2|Divisor function and integer partitions

From (2) and (3) is immediate the following recurrence relation:

$$nR(n) = \sum_{j=1}^{n} h(j) R(n-j), \quad h(j) = -\sum_{d/j} s(d) d, \quad h(0) = 0, \tag{6}$$

whose solution is given by [6]:

$$R(n) = \frac{1}{n!}\ B_n\Big(0!\ h(1),\ 1!\ h(2),\ 2!\ h(3),\ ...\ , (n-1)!\ h(n)\Big), \tag{7}$$

in terms of the complete Bell polynomials [6, 7], with the corresponding inversion [8] for $n \geq 1$:

$$(n-1)! \sum_{d/n} s(d) d = \sum_{k=1}^{n} (-1)^k (k-1)!\ B_{n,k}\Big(1!R(1),\ 2!R(2),\ ...\ , (n-k+1)!R(n-k+1)\Big), \tag{8}$$

with the presence of the partial Bell polynomials [8, 9]; besides, the coefficients in the expansion (1) can be calculated with the relation:

$$C_j(n) = \frac{(-1)^n}{n!}\ \Pi_{t=0}^{n-1}\Big(s(j)-t\Big), \quad n \geq 1, \quad C_j(0) = 1. \tag{9}$$

Now we shall apply our expressions to several sequences $\{s(n)\}$:

$$a).\ \ s(m) = \frac{1}{m} \quad \text{therefore} \quad \sum_{d/n} s(d) d = \sum_{d/n} 1 = d(n) = \text{number of divisors of n},$$

with (5) and (9) obtain the values:

$$C_j(1) = -\frac{1}{j}, \quad C_j(2) = \frac{1}{2j}\Big(\frac{1}{j}-1\Big), \quad C_j(3) = -\frac{1}{6j}\Big(\frac{1}{j}-1\Big)\Big(\frac{1}{j}-2\Big),\ ...\ ,$$

$$R(1) = -1, \quad R(2) = -\frac{1}{2}, \quad R(3) = \frac{1}{6}, \quad R(4) = -\frac{1}{24},\ ...\ , \tag{10}$$

then (8) generates the property:

$$(n-1)!\ d(n)=\sum_{k=1}^{n}(-1)^{k}(k-1)!\ B_{n,k}\Big(1!R(1),\ 2!R(2),\ \dots,(n-k+1)!R(n-k+1)\Big) \tag{11}$$

so, the number of divisors of an integer in terms of its partitions. This expression (11) is an alternative to the following relation deduced in [10]:

$$d(n)=\sum_{r=1}^{n}a_{n-r}\sum_{j=1}^{r}S_{r,j}, \tag{12}$$

where $S_{n,k}$ is the number of k's in all partitions of n, and [11]:

$$a_j=\begin{cases}0, & j\neq\frac{m}{2}(3m+1),\\ (-1)^m, & j=\frac{m}{2}(3m+1)\end{cases}\quad m\ =\ 0,\ \pm1,\ \pm2,\ \dots \tag{13}$$

that is:

$$a_j=\begin{cases}1, & j=0,5,7,22,26,51,57,92,100,145,155,\ \dots\\ -1, & j=1,2,12,15,35,40,70,77,117,126,176,\ \dots\\ 0, & \text{otherwise}\end{cases} \tag{14}$$

$$b).\ \ s(m)=1\ \text{ therefore }\ \sum_{d/n}s(d)d=\sigma(n)=\text{sum of divisors function},\ \ C_j(1)=-1, \tag{15}$$

$$C_j(t)=0,\ \ t\geq 2,$$

and from (2):

$$\sum_{j=0}^{\infty}R(j)q^j=\Pi_{n=1}^{\infty}(1-q^n)=(q:q)_\infty=\sum_{j=0}^{\infty}a(j)q^j\ \text{ therefore }\ R(n)=a_n, \tag{16}$$

then (5), (15) and (16) imply the identity:

$$a_n=\sum_{\lambda\vdash n}C_1(k_1)\ C_2(k_2)\ \dots\ C_n(k_n), \tag{17}$$

over all partitions without repeated parts. Besides, (6) gives the following recurrence relation obtained by Robbins [12] and Osler-Hassen-Chandrupatla [13]:

$$n\ a_n=-\sum_{j=1}^{n}\ \sigma(j)\ a_{n-j}, \tag{18}$$

and from (7) and (8):

$$a_n=\frac{1}{n!}\ B_n\Big(-0!\ \sigma(1),\ -1!\ \sigma(2),\ -2!\ \sigma(3),\ \dots,-(n-1)!\ \sigma(n)\Big), \tag{19}$$

$$(n-1)!\ \sigma_n=\sum_{k=1}^{n}(-1)^k(k-1)!\ B_{n,k}(1!\ a_1,\ 2!\ a_2,\ \dots,(n-k+1)!\ a_{n-k+1}).$$

c). $s(m)=\chi_4(m)/m$ where χ_4 is the nontrivial Dirichlet character (mod 4) [14, 15]:

$$\chi_4(n)=\begin{cases}(-1)^{(n-1)/2}, & n\text{ is odd },\\ 0\ , & n\text{ is even}\end{cases} \tag{20}$$

and we have the Jacobi's expression [16, 17, 18]:

$$\sum_{d/n}\ \chi_4(d)=\frac{1}{4}\ r_2(n), \tag{21}$$

involving the number of representations of n as the sum of two squares [14, 19, 20]; therefore:

$$C_j(1)=-\frac{\chi_4(j)}{j},\ C_2(j)=\frac{1}{2}\frac{\chi_4(j)}{j}\Big(\frac{\chi_4(j)}{j}\ -\ 1\Big),\ \dots,R(1)=-1,R(2)=0\ , \tag{22}$$

$$R(3)=-R(4)=\frac{1}{3}\ ,\ \dots$$

such that:

$$(n-1)!\, r_2(n) = 4\sum_{k=1}^{n}(-1)^k(k-1)!\, B_{n,k}\Big(1!R(1),\ 2!R(2),\ \dots,(n-k+1)!R(n-k+1)\Big). \tag{23}$$

d). $s(m) = \mu(m)/m$ participating the Möbius function [16, 17, 18, 21], therefore $\sum_{d/n}\ s(d)d = e_0(n)$ [22], with the Bellman's identity [12, 23, 24, 25, 26]:

$$\Pi_{j=1}^{\infty}\ (1-q^j)^{\frac{\mu(j)}{j}} = e^{-q} = \sum_{n=0}^{\infty} R(n)q^n \ \text{ therefore }\ R(j) = \frac{(-1)^j}{j!}, \tag{24}$$

and from (8):

$$\sum_{k=1}^{n}(-1)^k(k-1)!\, B_{n,k}\Big(-1,1,-1,1,\ \dots,(-1)^{n-k+1}\Big) = \begin{cases} 1, & n=1 \\ 0, & n\geq 2 \end{cases} \tag{25}$$

e). $s(m) = \varphi(m)/m$ involving the Euler's totient function $[16-18, 22, 27]$, therefore $\sum_{d/n}\ s(d)/d = n$ [22], with the property [12, 25]:

$$\Pi_{j=1}^{\infty}(1-q)^{\frac{\varphi(j)}{j}} = e^{-q/(1-q)} = \sum_{k=0}^{\infty} R(k)\, q^k \ \text{ therefore }\ R(n) = \frac{1}{n}\sum_{t=1}^{n}\binom{n}{k}\frac{(-1)^t}{(t-1)!}, \tag{26}$$

thus (8) implies the identity:

$$n! = \sum_{k=1}^{n}(-1)^k(k-1)!\, B_{n,k}\Big(1!\ R(1),\ 2!\ R(2),\ \dots,(n-k+1)!\ R(n-k+1)\Big), \tag{27}$$

such that $R(1) = -1$, $R(2) = -1/2$, $R(3) = -1/6$, $R(4) = 1/24$, $R(5) = 19/120$, ... It is possible to write $R(n)$ in terms of Kummer hypergeometric function or an associated Laguerre polynomial:

$$R(n) = -\ {}_1F_1(1-n;2;1) = -\frac{1}{n}\ L^1_{n-1}(1), \qquad n\geq 1. \tag{28}$$

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings.

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